Chapter 7

COMBINATIONS AND PERMUTATIONS

We have seen in the previous chapter that $(a + b)^n$ can be written as

$$\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n} b^n$$

where we have the specific formula for the binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2\cdot 3\ldots k}.$$

We now look at a different interpretation of these numbers and will see why $\binom{n}{k}$ is called "n choose k." Let

$$M = (1 + x_1)(1 + x_2)(1 + x_3).$$

Expanding this product, we get

$$M = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3.$$

The terms of this expansion correspond to the subsets of $S = \{x_1, x_2, x_3\}$. That is, we can associate the term 1 with the empty subset of S; the terms x_1, x_2, and x_3 with the singleton subsets of S; the terms x_1x_2, x_1x_3, and x_2x_3 with the doubleton subsets; and $x_1x_2x_3$ with S itself. (S is the only subset with 3 elements.)

Next we replace each of x_1, x_2, and x_3 by x in our two expressions for M. This results in

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

Thus we see that $\binom{3}{k}$ for $k = 0, 1, 2, 3$ is the number of ways of choosing a subset of k elements from a set S of 3 elements. Similarly, one can see that the number of ways of choosing k elements from a set of n elements is $\binom{n}{k}$.

For example, the set {1, 2, 3, 4, 5} with 5 elements has $\binom{5}{3}$ subsets having 3 elements. Since

$$\binom{5}{3} = \frac{5\cdot4\cdot3}{1\cdot2\cdot3} = 10,$$

it is not too difficult to write out all ten of these subsets as

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\},$$
$$\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}.$$

If we drop the braces enclosing the elements of each subset, the resulting sequence is said to be a **combination** of 3 things chosen from the set {1, 2, 3, 4, 5}. Thus the ten combinations of 3 things chosen from this set of 5 objects are

1, 2, 3;	1, 2, 4;	1, 2, 5;	1, 3, 4;	1, 3, 5;
1, 4, 5;	2, 3, 4;	2, 3, 5;	2, 4, 5;	3, 4, 5.

Note that changing the order in which the objects of a combination are written does not change the combination. For example, 1, 2, 4 is the same combination as 1, 4, 2.

The formula $\binom{n}{k} = \binom{n}{n-k}$ tells us that the entries on row n of the Pascal Triangle read the same left to right as they do right to left. The combinatorial significance of this formula is that the number of ways of choosing k elements from a set of n elements is equal to the number of ways of omitting n - k of the elements.

A problem analogous to that of finding the coefficients of a binomial expansion is that of finding the coefficents in

$$(x + y + z)^n.$$

These coefficients, called the **trinomial** **coefficients**, are naturally more complicated but, fortunately, can be expressed in terms of the binomial coefficients in the following way. Let us look for the coefficient of $x^6y^3z^1$ in $(x + y + z)^{10}$. From this product of ten factors we must choose six x's, three y's, and one z. We can choose six x's from a set of ten in $\binom{10}{6}$ ways and three

y's from the remaining four factors in $\binom{4}{3}$ ways, and the z from the remaining factor in

$\binom{1}{1}$ way. Therefore the trinomial coefficient of $x^6y^3z^1$ in $(x + y + z)^{10}$ can be written as

$\binom{10}{6}\binom{4}{3}\binom{1}{1}$, or, since $\binom{1}{1} = 1$, as $\binom{10}{6}\binom{4}{3}$. We can obtain an alternate

representation of this number, however, by choosing the z first in $\binom{10}{1}$ ways, then the six x's

from the remaining nine, and finally the three y's from the remaining three. Thus the coefficient would appear as $\binom{10}{1}\binom{9}{6}\binom{3}{3}$ or $\binom{10}{1}\binom{9}{6}$. Hence $\binom{10}{6}\binom{4}{3} = \binom{10}{1}\binom{9}{6}$. By choosing the y's first, one can see that this coefficient could also be expressed as
$\binom{10}{3}\binom{7}{6}$ or $\binom{10}{3}\binom{7}{1}$. The reader may find other forms of the coefficient.

This problem can be generalized similarly to find the coefficients of $(x + y + z + \cdots + w)^n$; they are called **multinomial** **coefficients**. It can readily be shown that the coefficient of $x^a y^b z^c \ldots w^d$ in the expansion of $(x + y + z + \cdots + w)^n$ is

$$\frac{n!}{a!b!c!\ldots d!}$$

where, of course, the sum $a + b + c + \cdots + d$ of the exponents must be n.

Another interesting problem, and one with frequent applications, is that of finding the number of ways in which one can arrange a set of objects in a row, that is, the number of **permutations** of the set. Let us consider the set of four objects *a, b, c, d.* They can be arranged in the following ways:

a b c d	*b a c d*	*c a b d*	*d a b c*
a b d c	*b a d c*	*c a d b*	*d a c b*
a c b d	*b c a d*	*c b a d*	*d b a c*
a c d b	*b c d a*	*c b d a*	*d b c a*
a d b c	*b d a c*	*c d a b*	*d c a b*
a d c b	*b d c a*	*c d b a*	*d c b a*

Rather than write them all out, if we are only interested in the number of arrangements, we may think of the problem thus: We have four spaces to fill. If we put, for example, the *b* in the first, we have only the *a, c,* and *d* to choose from in filling the remaining three. And if we put the *d* in the second, we have only *a* and *c* for the remaining; and so forth. So we have four choices for the first space, three for the second, two for the third, and one for the fourth. This gives us $4 \cdot 3 \cdot 2 \cdot 1$, or 4! arrangements of four objects. This argument can be used to show that there are $n!$ arrangements of n objects.

We may also consider the possibility of arranging, in a row, r objects chosen from a set of n. We have n choices for the first space, $n - 1$ for the second, $n - 2$ for the third, and so on. Finally we have $n - r + 1$ choices for the rth space, giving a total of $n(n - 1)(n - 2)\ldots(n - r + 1)$ possible arrangements (or permutations). This can be written in terms of factorials as follows:

$$
\begin{aligned}
&n(n-1)(n-2)\ldots(n-r+1) \\
&\quad = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1} \\
&\quad = \frac{n!}{(n-r)!}.
\end{aligned}
$$

It should be noted that this is not the number of combinations of r objects taken from a set of n, since in permutations order is important; in combinations it is not. For example, if we consider the three objects *a, b*, and *c*, the number of permutations of two objects chosen from them is $3 \cdot 2$, the arrangements *ab, ba, bc, cb, ca, ac*. However, the number of combinations is $\frac{3!}{2!1!} = 3$, and the combinations are *a* and *b, b* and *c*, and *a* and *c*.

Next we define even and odd permutations of 1, 2, ..., n; this topic is used in Chapter 9 and in higher algebra.

We begin with the case $n = 3$, that is the numbers 1, 2, 3. With each permutation

$$i, j, k$$

of these three numbers, we associate the product of differences

$$p = (j - i)(k - i)(k - j).$$

If p is positive, the associated permutation is called **<u>even</u>**; if p is negative, the associated permutation is **<u>odd</u>**. Three of the 3! permutations of 1, 2, 3 are even and three are odd. The even ones are listed in the first column, and the odd ones in the second column:

1, 2, 3	1, 3, 2
2, 3, 1	2, 1, 3
3, 1, 2	3, 2, 1

For general n, a permutation $i, j, h, k, \ldots, r, s$ of 1, 2, 3, ... , n is associated with the product

$$p = [(j - i)][(h - i)(h - j)][(k - i)(k - j)(k - h)] \ldots [(s - i)(s - j)(s - h)(s - k)\ldots(s - r)]$$

of all the differences of two of $i, j, h, k, \ldots, r, s$ in which the number that appears first is subtracted from the other. If the permutation $i, j, h, k, \ldots, r, s$ is written in the notation $a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n$, then the product p takes the form

$$p = [(a_2 - a_1)][(a_3 - a_1)(a_3 - a_2)][(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)]\ldots [(a_n - a_1)(a_n - a_2)(a_n - a_3)(a_n - a_4)\ldots(a_n - a_{n-1})].$$

If the product p is positive, the permutation is **<u>even</u>**; if p is negative, the permutation is **<u>odd</u>**.

Problems for Chapter 7

1. Write out all the combinations of two letters chosen from *a, b, c, d,* and *e*.

2. Write out all the combinations of three letters chosen from *a, b, c, d,* and *e*.

3. Write out all the permutations of two letters chosen from *a, b, c, d,* and *e*.

4. Write out all the permutations of three letters chosen from *a, b, c, d,* and *e*.

5. Find the positive integer that is the coefficient of $x^3y^7z^2$ in $(x + y + z)^{12}$.

6. Express the trinomial coefficient of the previous problem in six ways as a product of two binomial coefficients.

7. How many combinations are there of 1, 2, 3, or 4 elements from a set of 5 elements?

8. How many non-empty proper subsets are there of a set of n elements? That is, how many combinations are there of 1, 2, ... , or n - 1 elements?

9. Express the coefficient of $x^3y^7w^2$ in $(x + y + z + w)^{12}$ in six different ways as a product of two binomial coefficients.

10. Express the coefficient of $x^2y^3z^4w^2$ in $(x + y + z + w)^{11}$ in six different ways as a product of three binomial coefficients.

11. Find the coefficient of $x^2y^9z^3w$ in $(2x + y - z + w)^{15}$.

12. Find the coefficient of x^ryzw in $(x + y + z + w)^{r+3}$.

13. Show that $x^5y^2z^9$ has the same coefficient in $(x + y + z + w)^{16}$ as in $(x + y + z)^{16}$.

14. What is the relation between the coefficient of xy^7z^2 in $(x + y - z)^{10}$ and its coefficient in $(x - y + z + w)^{10}$?

15. Let a, b, and n be positive integers, with $n > a + b$. Show that

$$\binom{n}{a}\binom{n-a}{b} + \binom{n}{b}\binom{n-b}{a-1} + \binom{n}{a}\binom{n-a}{b-1} = \binom{n+1}{a}\binom{n-a+1}{b}.$$

16. Express the coefficient of $x^2y^4z^6$ in $(x + y + z)^{12}$ as the sum of three of the trinomial coefficients in the expansion of $(x + y + z)^{11}$.

17. What is the sum of all the trinomial coefficients in $(x + y + z)^{100}$?

18. What is the sum of the coefficients in each of the following:

(a) $(x + y - z)^{100}$?

(b) $(x - y + z - w)^{100}$?

19. List the even permutations of 1, 2, 3, 4.

20. List the odd permutations of 1, 2, 3, 4.

R 21. Let P be a permutation $i, j, h, \ldots k$ of $1, 2, 3, \ldots, n$.

(a) Show that if i and j are interchanged, P changes from odd to even or from even to odd.

(b) Show that if any two adjacent terms in P are interchanged, P changes from odd to even or from even to odd.

(c) Show that the interchange of any two terms in P can be considered to be the result of an odd number of interchanges of adjacent terms.

(d) Show that if any two terms in the permutation P are interchanged, P changes from odd to even or from even to odd.

(e) Given that $n \geq 2$, show that half of the permutations of 1, 2, ... , n are even and half are odd, that is, that there are $\frac{n!}{2}$ even permutations and the same number of odd ones.

R 22. (a) Let P be a permutation a, b, c, d of the numbers 1, 2, 3, 4. Let $d = 4$ and let Q be the associated permutation a, b, c of 1, 2, 3. Show that P and Q are either both even or both odd.

(b) Let R be a permutation $i, j, ..., h, n$ of the numbers 1, 2, ..., n-1, n in which the last term of R is n. Let S be the associated permutation $i, j, ..., h$ of 1, 2, ..., n-1 obtained by dropping the last term of R. Show that R and S are either both even or both odd.

23. How many triples of positive integers r, s, and t are there with $r < s < t$ and:

(a) $r + s + t = 52$?

(b) $r + s + t = 352$?

24. The arrangement $\begin{bmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & 8 \end{bmatrix}$ has the property that the numbers increase as one goes down or to the right.

(a) How many other arrangements are there of the numbers 1, 2, ..., 8 in 2 rows and 4 columns with this property?

(b) How many arrangements are there of the numbers 1, 2, ..., 14 in 2 rows and 7 columns with this property?

(c) How many arrangements are there of the numbers 1, 2, ..., 12 in 3 rows and 4 colunms with this property?